

GENERATING ALL WIGNER FUNCTIONS

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*zachos@hep.anl.gov***Abstract**

In the context of phase-space quantization, matrix elements and observables result from integration of c-number functions over phase space, with Wigner functions serving as the quasi-probability measure. The complete sets of Wigner functions necessary to expand all phase-space functions include off-diagonal Wigner functions, which may appear technically involved. Nevertheless, it is shown here that suitable generating functions of these complete sets can often be constructed, which are relatively simple, and lead to compact evaluations of matrix elements. New features of such generating functions are detailed and explored for integer-indexed sets, such as for the harmonic oscillator, as well as continuously indexed ones, such as for the linear potential and the Liouville potential. The utility of such generating functions is illustrated in the computation of star functions, spectra, and perturbation theory in phase space.

1 Introduction

General phase-space functions $f(x, p)$ and $g(x, p)$ compose noncommutatively through Groenewold's \star -product [1], which is the unique associative pseudodifferential deformation [2] of ordinary products:

$$\star \equiv e^{i\hbar(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)/2} . \quad (1)$$

This product is the cornerstone of deformation (phase-space) quantization [3, 2, 4, 5], as well as applications of matrix models and non-commutative geometry ideas in M-physics [6]. Its mechanics, however, is not always straightforward.

The practical Fourier representation of this product as an integral kernel has been utilized widely since Baker's [7] early work,

$$f \star g = \frac{1}{\hbar^2 \pi^2} \int dp' dp'' dx' dx'' f(x', p') g(x'', p'') \exp \left(\frac{-2i}{\hbar} (p(x' - x'') + p'(x'' - x) + p''(x - x')) \right) . \quad (2)$$

The determinantal nature of the star product controls the properties of the phase-space trace [8, 9],

$$\int dpdx f \star g = \int dpdx fg = \int dpdx g \star f. \quad (3)$$

The above \star -product and phase-space integrals provide the multiplication law and, respectively, the trace in phase-space quantization [3], the third autonomous and logically complete formulation of quantum mechanics beyond the conventional formulations based on operators in Hilbert space or path integrals. (This formulation is reviewed in [2, 5].) Properly ordered operators (e.g., Weyl-ordered) correspond uniquely to phase-space c-number functions (referred to as “classical kernels” of the operators in question); operator products correspond to \star -products of their classical kernels; and operator matrix elements, conventionally consisting of traces thereof with the density matrix, correspond to phase-space integrals of the classical kernels with the Wigner function (WF), the Weyl correspondent of the density matrix [10, 5]. The celebrated \star -genvalue functional equations determining the Wigner functions [8, 11] and their spectral properties (e.g. projective orthogonality [12]) are reviewed and illustrated in [4].

The functions introduced by Wigner [10] and Szilard correspond to diagonal elements of the density matrix, but quantum mechanical applications (such as perturbation theory), as well as applications in noncommutative soliton problems [13] often require the evaluation of off-diagonal matrix elements; they therefore utilize the complete set of diagonal and off-diagonal generalized Wigner functions introduced by Moyal [3]. For instance, in noncommutative soliton theory, the diagonal WFs are only complete for radial phase-space functions (functions \star -commuting with the harmonic oscillator hamiltonian—the radius squared), whereas deviations from radial symmetry necessitate the complete off diagonal set.

As for any representation problem, the particular features of the \star -equations under consideration frequently favor an optimal basis of WFs; but, even in the case of the oscillator, the equations are technically demanding. It is pointed out here, however, that suitable generating functions for them, acting as a transform of these basis sets, often result in substantially simpler and more compact objects, which are much easier to use, manipulate, and intuit. Below, after some elementary overview of the Weyl correspondence formalism (Sec 2), we illustrate such functions for the harmonic oscillator (Sec 3), which serves as the archetype of WF bases indexed discretely; it turns out that these generating functions amount to the phase-space coherent states for WFs, and also the WFs of coherent state wavefunctions (Appendix A). Direct applications to first order perturbation theory are illustrated in Appendix B.

For sets indexed continuously, the generating function may range from a mere Fourier transform, illustrated by the linear potential (Sec 4), to a less trivial continuous transform we provide for the Liouville potential problem (Sec 5), where the advantage of the transform method comes to cogent evidence.

Throughout our discussion, we provide the typical \star -composition laws of such generating functions, as well as applications such as the evaluation of \star -exponentials of phase-space functions (Appendix C), or \star -versions of modified Bessel functions (technical aspects of integral transforms of which are detailed in Appendix D). Appendix E provides the operator (Weyl-) correspondent to the generating function for the Liouville diagonal WF introduced in Sec 5.

2 Overview of General Relations in the Weyl representation

Without loss of generality, we review basic concepts in two-dimensional phase space, (x, p) , as the extension to higher dimensions is straightforward. In addition, we first address discrete spectra, E_n , $n = 0, 1, 2, 3, \dots$, and will only later generalize to continuous spectra.

In the Weyl correspondence [14], c-number phase-space kernels $a(x, p)$ of suitably ordered operators $\mathcal{A}(\mathcal{X}, \mathcal{P})$ are defined by

$$a(x, p) \equiv \frac{1}{2\pi} \int dy e^{-iyp} \langle x - \frac{\hbar}{2}y | \mathcal{A}(\mathcal{X}, \mathcal{P}) | x + \frac{\hbar}{2}y \rangle. \quad (4)$$

Conversely, the ordering of these operators is specified through

$$\mathcal{A}(\mathcal{X}, \mathcal{P}) = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp a(x, p) \exp(i\tau(\mathcal{P} - p) + i\sigma(\mathcal{X} - x)). \quad (5)$$

An operator product then corresponds to a star-composition of these kernels [1],

$$a(x, p) \star b(x, p) = \frac{1}{2\pi} \int dy e^{-iyp} \langle x - \frac{\hbar}{2}y | \mathcal{A}(\mathcal{X}, \mathcal{P}) \mathcal{B}(\mathcal{X}, \mathcal{P}) | x + \frac{\hbar}{2}y \rangle, \quad (6)$$

Moyal [3] appreciated that the density matrix in this phase-space representation is a hermitean generalization of the Wigner function:

$$\begin{aligned} f_{mn}(x, p) &\equiv \frac{1}{2\pi} \int dy e^{-iyp} \langle x - \frac{\hbar}{2}y | \psi_n \rangle \langle \psi_m | x + \frac{\hbar}{2}y \rangle \\ &= \frac{1}{2\pi} \int dy e^{-iyp} \psi_m^*(x - \frac{\hbar}{2}y) \psi_n(x + \frac{\hbar}{2}y) = f_{nm}^*(x, p), \end{aligned} \quad (7)$$

where the $\psi_m(x)$ s are (ortho-)normalized solutions of a Schrödinger problem. (Wigner [10] mainly considered the diagonal elements of the density matrix (pure states), usually denoted as $f_m \equiv f_{mm}$.) As a consequence, matrix elements of operators are produced by mere phase space integrals [3],

$$\langle \psi_m | \mathcal{A} | \psi_n \rangle = \int dx dp a(x, p) f_{mn}(x, p). \quad (8)$$

The standard machinery of density matrices then is readily transcribed in this language, e.g. the trace relation [3],

$$\int dx dp f_{mn}(x, p) = \int dx \psi_n^*(x) \psi_m(x) = \delta_{mn}; \quad (9)$$

and [8]

$$f_{mn} \star f_{kl} = \frac{1}{2\pi\hbar} \delta_{ml} f_{kn} = \frac{1}{\hbar} \delta_{ml} f_{kn}. \quad (10)$$

Given (3), it follows from the above that [3]

$$\int dx dp f_{mn}(x, p) f_{lk}^*(x, p) = \frac{1}{2\pi\hbar} \delta_{ml} \delta_{nk}. \quad (11)$$

For complete sets of input wavefunctions, it also follows that [3]:

$$\sum_{m,n} f_{mn}(x, p) f_{mn}^*(x', p') = \frac{1}{2\pi\hbar} \delta(x - x') \delta(p - p'). \quad (12)$$

An arbitrary phase-space function $\varphi(x, p)$ can thus be expanded as

$$\varphi(x, p) = \sum_{m,n} c_{mn} f_{mn}(x, p), \quad (13)$$

the coefficients being specified through (11),

$$c_{mn} = 2\pi\hbar \int dx dp f_{mn}^*(x, p) \varphi(x, p). \quad (14)$$

Further note the resolution of the identity [3],

$$\sum_n f_{nn}(x, p) = \frac{1}{2\pi\hbar} = \frac{1}{h}. \quad (15)$$

For instance, for eigenfunctions of the hamiltonian $\mathcal{H}(\mathcal{X}, \mathcal{P})$ with eigenvalues E_n , the corresponding WFs satisfy the following star-eigenvalue equations [8] (further cf [11, 4]), with $H(x, p)$, the phase-space kernel of $\mathcal{H}(\mathcal{X}, \mathcal{P})$:

$$H \star f_{mn} = E_n f_{mn}, \quad f_{mn} \star H = E_m f_{mn}. \quad (16)$$

The time dependence of a pure state WF is given by Moyal's dynamical equation [3]:

$$i\hbar \frac{\partial}{\partial t} f(x, p; t) = H \star f(x, p; t) - f(x, p; t) \star H. \quad (17)$$

By virtue of the \star -unitary evolution operator (a “ \star -exponential” [2]),

$$U_\star(x, p; t) = e_\star^{itH/\hbar} \equiv 1 + (it/\hbar)H(x, p) + \frac{(it/\hbar)^2}{2!}H \star H + \frac{(it/\hbar)^3}{3!}H \star H \star H + \dots, \quad (18)$$

the time-evolved WF is obtained formally in terms of the WF at $t = 0$,

$$f(x, p; t) = U_\star^{-1}(x, p; t) \star f(x, p; 0) \star U_\star(x, p; t). \quad (19)$$

(These associative combinatoric operations completely parallel those of operators in the conventional formulation of quantum mechanics in Hilbert space [15].) Just like any star-function of H , this \star -exponential can be computed, [16]

$$\exp_\star(itH/\hbar) = \exp_\star(itH/\hbar) \star 1 = \exp_\star(itH/\hbar) \star 2\pi\hbar \sum_n f_{nn} = 2\pi\hbar \sum_n e^{itE_n/\hbar} f_{nn}. \quad (20)$$

(Of course, for $t = 0$, the obvious identity resolution is recovered).

For continuous spectra, the sums in the above relations extend to integrals over a continuous parameter (the energy), and the Kronecker δ_{mn} s into δ -functions (these last ones reflecting the infinite normalizations of unnormalizable states). E.g., eqns (9, 11) extend to

$$\int dx dp f_{E_1 E_2}(x, p) = \delta(E_1 - E_2), \quad (21)$$

$$\int dx dp f_{E_1 E_2}(x, p) f_{E'_1 E'_2}^*(x, p) = \frac{1}{2\pi\hbar} \delta(E_1 - E'_1) \delta(E_2 - E'_2), \quad (22)$$

Completeness (12) extends to

$$\int dE_1 dE_2 f_{E_1 E_2}(x, p) f_{E_1 E_2}^*(x', p') = \frac{1}{2\pi\hbar} \delta(x - x') \delta(p - p'). \quad (23)$$

More generally, (10) extends to

$$f_{E_1 E_2} \star f_{E'_1 E'_2} = \frac{1}{2\pi\hbar} \delta(E_1 - E'_2) f_{E'_1 E_2}. \quad (24)$$

Finally, eqn (15) extends to

$$\begin{aligned} \frac{1}{2\pi\hbar} &= \frac{1}{2\pi} \int dy e^{-ipy} \int dE \langle x - \frac{\hbar y}{2} | E \rangle \langle E | x + \frac{\hbar y}{2} \rangle \\ &= \int dE f_{EE}(x, p), \end{aligned} \quad (25)$$

and hence (20) extends to

$$\exp_\star(itH/\hbar) = 2\pi\hbar \int dE e^{itE/\hbar} f_{EE}(x, p). \quad (26)$$

3 Generating Functions for the Harmonic Oscillator

Consider the harmonic oscillator,

$$H(x, p) = \frac{1}{2}(p^2 + x^2), \quad (27)$$

where, without loss of generality, parameters have been absorbed in the phase space variables: $m = 1$, $\omega = 1$. Further recall that the normalized eigenfunctions of the corresponding operator hamiltonian \mathcal{H} are $\psi_n(x) = (\sqrt{\pi} 2^n n!)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} H_n(x)$, for the eigenvalues $E_n = \hbar(n + 1/2)$. Define a radial and an angular variable,

$$z \equiv 4H = 2(x^2 + p^2), \quad \tan \theta = \frac{p}{x}, \quad (28)$$

so that

$$a\sqrt{2} \equiv (x + ip) = |x + ip| e^{i\theta} = \left(\frac{z}{2}\right)^{\frac{1}{2}} e^{i\theta}. \quad (29)$$

Groenewold [1], as well as Bartlett and Moyal [17], have worked out the complete sets of solutions to Moyal's time-evolution equation (17), which are all linear combinations of terms $\exp(it(m - n)) f_{mn}$. They solved that equation indirectly, by evaluating the integrals (7) for time-dependent Hermite wavefunctions, which yield generalized Laguerre polynomial-based functions. More directly, Fairlie [8] dramatically simplified the derivation of the solution by relying on his fundamental equation (16). He thus confirmed Groenewold's WFs [1, 17],

$$f_{mn}(x, p) = \frac{(-1)^m}{\pi} \sqrt{\frac{m!}{n!}} z^{\frac{n-m}{2}} e^{-z/2} e^{i(n-m)\theta} L_m^{n-m}(z). \quad (30)$$

The special case of diagonal elements,

$$f_n \equiv f_{nn} = \frac{(-1)^n}{\pi} e^{-z/2} L_n(z), \quad (31)$$

constitutes the time-independent “ \star -genfunctions” of the oscillator hamiltonian kernel [4] (i.e. the complete set of solutions of the time-independent Moyal equation $H \star f - f \star H = 0$, where $H \star f_n = E_n f_n$. Incidentally, (10) restricted to diagonal WFs closes them under \star -multiplication [12], $f_m \star f_n = \delta_{mn} f_m / (2\pi\hbar)$.) That is to say, “radially symmetric” phase-space functions, i.e. functions that only depend on z but not θ , can be expanded in terms of merely these diagonal elements—unlike the most general functions in phase space which require the entire set of off-diagonal f_{mn} above for a complete basis. Note, however, that all \star -products of such radially symmetric functions are commutative, since, manifestly,

$$\sum_n c_n f_n \star \sum_m d_m f_m = \sum_m d_m f_m \star \sum_n c_n f_n. \quad (32)$$

Moreover, the \star -exponential (20) for this set of \star -genfunctions is directly seen to amount to

$$\exp_\star(itH/\hbar) = \left(\cos\left(\frac{t}{2}\right)\right)^{-1} \exp\left(\frac{2i}{\hbar} H \tan\left(\frac{t}{2}\right)\right), \quad (33)$$

which is, to say, a Gaussian in phase space [2]. As an application, note that the hyperbolic tangent \star -composition law of gaussians follows trivially (since these amount to \star -exponentials with additive time intervals, $\exp_\star(tf) \star \exp_\star(Tf) = \exp_\star((t+T)f)$, [2],

$$\exp\left(-\frac{a}{\hbar}(x^2 + p^2)\right) \star \exp\left(-\frac{b}{\hbar}(x^2 + p^2)\right) = \frac{1}{1+ab} \exp\left(-\frac{a+b}{\hbar(1+ab)}(x^2 + p^2)\right). \quad (34)$$

We now introduce the following generating function for the entire set of generalized Wigner functions,

$$\begin{aligned} G(\alpha, \beta; x, p) &\equiv \sum_{m,n} \frac{\alpha^m}{\sqrt{m!}} \frac{\beta^n}{\sqrt{n!}} f_{mn} \\ &= \frac{1}{\pi} \sum_n \beta^n \frac{1}{n!} z^{n/2} e^{-z/2} e^{in\theta} \sum_m \left(-z^{-\frac{1}{2}} e^{-i\theta} \alpha\right)^m L_m^{n-m}(z). \end{aligned} \quad (35)$$

Utilizing the identity [18] 8.975.2,

$$\sum_{m=0}^{\infty} L_m^{n-m}(z) k^m = e^{-zk} (1+k)^n, \quad (36)$$

we obtain

$$\begin{aligned} G(\alpha, \beta; x, p) &= \frac{1}{\pi} e^{-z/2} \sum_n \frac{1}{n!} \left(\beta \sqrt{z} e^{i\theta}\right)^n e^{-z(-z^{-\frac{1}{2}} e^{-i\theta} \alpha)} \left(1 - z^{-\frac{1}{2}} e^{-i\theta} \alpha\right)^n \\ &= \frac{1}{\pi} e^{-z/2} \sum_n \frac{1}{n!} \left(\beta \sqrt{z} e^{i\theta} - \alpha \beta\right)^n e^{\sqrt{z} e^{-i\theta} \alpha} \\ &= \frac{1}{\pi} e^{-z/2} e^{\beta \sqrt{z} e^{i\theta} - \alpha \beta} e^{\sqrt{z} e^{-i\theta} \alpha}. \end{aligned} \quad (37)$$

Thus,

$$G(\alpha, \beta; x, p) = \frac{1}{\pi} \exp\left(\sqrt{z}(\alpha e^{-i\theta} + \beta e^{i\theta}) - \alpha \beta - \frac{z}{2}\right). \quad (38)$$

Since

$$\sqrt{z}(\alpha e^{-i\theta} + \beta e^{i\theta}) = \sqrt{2}(\alpha + \beta)x - \sqrt{2}ip(\alpha - \beta), \quad (39)$$

one can re-express:

$$G(\alpha, \beta; x, p) = G^*(\beta, \alpha; x, p) = \frac{1}{\pi} \exp \left(\alpha\beta - \left(x - \frac{\alpha + \beta}{\sqrt{2}} \right)^2 - \left(p + i \frac{\alpha - \beta}{\sqrt{2}} \right)^2 \right). \quad (40)$$

As the name implies, from $G(\alpha, \beta; x, p)$, the f_{mn} s are generated by

$$f_{mn}(x, p) = \frac{1}{\sqrt{m!n!}} \frac{\partial^m}{\partial \alpha^m} \frac{\partial^n}{\partial \beta^n} G(\alpha, \beta; x, p) \Big|_{\alpha=\beta=0}. \quad (41)$$

These functions \star -compose as

$$G(\alpha, \beta) \star G(\epsilon, \zeta) = \frac{e^{\alpha\zeta}}{2\pi\hbar} G(\epsilon, \beta). \quad (42)$$

The phase-space trace is

$$\int dx dp G(\alpha, \beta) = e^{\alpha\beta}. \quad (43)$$

By (16), the action of the Hamiltonian kernel on this function is

$$H \star G = \hbar \left(\frac{1}{2} + \beta \frac{\partial}{\partial \beta} \right) G = \hbar \left(\frac{1}{2} - \alpha\beta + \beta\sqrt{z}e^{i\theta} \right) G, \quad (44)$$

and

$$G \star H = \hbar \left(\frac{1}{2} + \alpha \frac{\partial}{\partial \alpha} \right) G = \hbar \left(\frac{1}{2} - \alpha\beta + \alpha\sqrt{z}e^{-i\theta} \right) G. \quad (45)$$

Consequently,

$$\int dx dp H \star G(\alpha, \beta) = \hbar \left(\frac{1}{2} + \beta \frac{\partial}{\partial \beta} \right) e^{\alpha\beta} = \hbar \left(\frac{1}{2} + \alpha\beta \right) e^{\alpha\beta}. \quad (46)$$

The spectrum then follows by operating on both sides of this equation,

$$\begin{aligned} E_n &= \frac{1}{n!} \frac{\partial^n}{\partial \alpha^n} \frac{\partial^n}{\partial \beta^n} \int dx dp H \star G(\alpha, \beta) \Big|_{\alpha=\beta=0} \\ &= \frac{\hbar}{n!} \frac{\partial^n}{\partial \alpha^n} \frac{\partial^n}{\partial \beta^n} \left(\frac{1}{2} + \alpha\beta \right) e^{\alpha\beta} \Big|_{\alpha=\beta=0} = \hbar \left(\frac{1}{2} + n \right). \end{aligned} \quad (47)$$

In general, matrix elements of operators may be summarized compactly through this generating function in phase space.

This generating function could be interpreted as a phase-space coherent state, or the off-diagonal WF of coherent states, as discussed in Appendix A, [19],

$$G(\alpha, \beta; x, p) = \exp_{\star}(\beta a^{\dagger}) f_0 \exp_{\star}(\alpha a). \quad (48)$$

$$\begin{aligned}
a \star G(\alpha, \beta) &= \hbar \beta G(\alpha, \beta), & a^\dagger \star G(\alpha, \beta) &= \frac{\partial}{\partial \beta} G(\alpha, \beta) \\
G(\alpha, \beta) \star a &= \frac{\partial}{\partial \alpha} G(\alpha, \beta), & G(\alpha, \beta) \star a^\dagger &= \hbar \alpha G(\alpha, \beta),
\end{aligned} \tag{49}$$

and hence eqs (44,45) amount to

$$\begin{aligned}
H \star G(\alpha, \beta) &= (a^\dagger \star a + \frac{\hbar}{2}) \star G(\alpha, \beta) = \hbar \left(\beta \frac{\partial}{\partial \beta} + \frac{1}{2} \right) G(\alpha, \beta) \\
G(\alpha, \beta) \star H &= G(\alpha, \beta) \star (a^\dagger \star a + \frac{\hbar}{2}) = \hbar \left(\alpha \frac{\partial}{\partial \alpha} + \frac{1}{2} \right) G(\alpha, \beta).
\end{aligned} \tag{50}$$

This formalism finds application in, e.g., perturbation theory in phase space, cf. Appendix B.

4 Generating Functions for the Linear Potential

The linear potential in phase space has been addressed [11] (also see [19, 4]). We shall adopt the simplified conventions of [4], ie. $m = 1/2$, $\hbar = 1$. The Hamiltonian kernel is then,

$$H(x, p) = p^2 + x, \tag{51}$$

and the eigenfunctions of \mathcal{H} are Airy functions,

$$\psi_E(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dX e^{iX(E-x-X^2/3)} = \text{Ai}(x-E), \tag{52}$$

indexed by the continuous energy E . The spectrum being continuous, the Airy functions are not square integrable, but have continuum normalization, $\int dx \psi_{E_1}^*(x) \psi_{E_2}(x) = \delta(E_1 - E_2)$, instead. Thus, (21) et seq. are now operative. The generalized WFs are [11]

$$\begin{aligned}
f_{E_1 E_2}(x, p) &= \frac{1}{4\pi^2} \int dz e^{iz(\frac{E_1+E_2}{2}-x-p^2-z^2/12)} e^{ip(E_1-E_2)} \\
&= e^{ip(E_1-E_2)} \frac{2^{2/3}}{2\pi} \text{Ai} \left(2^{2/3} \left(x + p^2 - \frac{E_1 + E_2}{2} \right) \right).
\end{aligned} \tag{53}$$

The \star -exponential (26) then is again a plain exponential of the shifted hamiltonian kernel,

$$\exp_\star(it(x+p^2)) = 2\pi \int_{-\infty}^{\infty} dE e^{iEt} \frac{2^{2/3}}{2\pi} \text{Ai} \left(2^{2/3} \left(x + p^2 - \frac{E_1 + E_2}{2} \right) \right) = \exp(it(x+p^2+t^2/12)). \tag{54}$$

(This could also be derived directly, as the CBH expansion simplifies dramatically in this case, cf. Appendix C.) As before, the \star -composition law for plain exponentials of the hamiltonian kernel function follows,

$$\exp(a(x+p^2)) \star \exp(b(x+p^2)) = \exp \left((a+b) \left(x + p^2 - \frac{1}{4}ab \right) \right). \tag{55}$$

Since the complete basis Wigner functions are now indexed continuously, a generating function from them must rely on an integral instead of an infinite sum. The simplest transform is possibly a double Fourier

transform with respect to the energy indices (but note the transform factors $\exp(iE_1X)$, $\exp(-iE_2Y)$ may also be regarded as plane waves). Suitably normalized,

$$\begin{aligned}
G(X, Y; x, p) &\equiv 2\pi \int_{-\infty}^{+\infty} dE_1 \int_{-\infty}^{+\infty} dE_2 \left(\frac{1}{\sqrt{2\pi}} e^{iE_1X} \right) f_{E_1E_2}(x, p) \left(\frac{1}{\sqrt{2\pi}} e^{-iE_2Y} \right) \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dE_1 \int_{-\infty}^{+\infty} dE_2 e^{i(E_1-E_2)p+iE_1X-iE_2Y} 2^{2/3} \text{Ai} \left(2^{2/3} \left(x + p^2 - \frac{E_1+E_2}{2} \right) \right) \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dE \int_{-\infty}^{+\infty} d\omega e^{i\omega p+i(E+\omega/2)X-i(E-\omega/2)Y} 2^{2/3} \text{Ai} \left(2^{2/3} (x + p^2 - E) \right) \\
&= \delta \left(p + \frac{X+Y}{2} \right) \int_{-\infty}^{+\infty} dE e^{iE(X-Y)} 2^{2/3} \text{Ai} \left(2^{2/3} (x + p^2 - E) \right) \\
&= \delta \left(p + \frac{X+Y}{2} \right) \int_{-\infty}^{+\infty} dE e^{iE(X-Y)} \frac{1}{2\pi} \int dz e^{iz(E-x-p^2-z^2/12)} \\
&= \delta \left(p + \frac{X+Y}{2} \right) e^{i(X-Y)(x+p^2+(X-Y)^2/12)}.
\end{aligned} \tag{56}$$

The phase-space trace is

$$\int dx dp G(X, Y; x, p) = 2\pi \delta(X - Y), \tag{57}$$

and, given (24) for these functions, $f_{E_1E_2} \star f_{E'_1E'_2} = \frac{1}{2\pi} \delta(E_1 - E'_2) f_{E'_1E_2}$, the \star -composition law for these G s is

$$G(X, Y; x, p) \star G(W, Z; x, p) = \delta(X - Z) G(W, Y; x, p). \tag{58}$$

5 Generating Functions for the Liouville Potential

A less trivial system with a continuous spectrum is the Hamiltonian with the Liouville potential, [20, 21]. In the conventions of [4], ($\hbar = 1$, $m = 1/2$), the Hamiltonian kernel is

$$H = p^2 + e^{2x}, \tag{59}$$

and the eigenfunctions of the corresponding \mathcal{H} are

$$\psi_E(x) = \psi_E^*(x) = \frac{1}{\pi} \sqrt{\sinh(\pi\sqrt{E})} K_{i\sqrt{E}}(e^x), \tag{60}$$

with continuum normalizations $\int dx \psi_{E_1}^*(x) \psi_{E_2}(x) = \delta(E_1 - E_2)$. The modified Bessel function (Cf. [22], Ch VI, §6.22) can be written in the Heine-Schl fli form,

$$K_{ip}(e^x) = \frac{1}{2} \int_{-\infty}^{\infty} dX \exp(-e^x \cosh X + iXp) = K_{-ip}(e^x). \tag{61}$$

The non-diagonal WF is then

$$f_{E_1E_2}(x, p) = \frac{1}{\pi^3} \int dy e^{-2ipy} \sqrt{\sinh(\pi\sqrt{E_1})} K_{i\sqrt{E_1}}^*(e^{x-y}) \sqrt{\sinh(\pi\sqrt{E_2})} K_{i\sqrt{E_2}}(e^{x+y}). \tag{62}$$

This Wigner function amounts to Meijer's G function,

$$f_{E_1 E_2}(x, p) = \frac{1}{8\pi^3} \sqrt{\sinh(\pi\sqrt{E_1}) \sinh(\pi\sqrt{E_2})} G_{04}^{40} \left(\frac{e^{4x}}{16} \left| \frac{ip + i\sqrt{E_1}}{2}, \frac{ip - i\sqrt{E_1}}{2}, \frac{-ip + i\sqrt{E_2}}{2}, \frac{-ip - i\sqrt{E_2}}{2} \right. \right). \quad (63)$$

Alternatively, the WF may be written as a double integral representation,

$$\begin{aligned} f_{E(k) E(q)}(x, p) &= \\ &= \frac{1}{2\pi^3} \sqrt{\sinh(\pi\sqrt{E(k)}) \sinh(\pi\sqrt{E(q)})} \int dX dY e^{ikX} e^{iqY} \left(\frac{\cosh Y}{\cosh X} \right)^{ip} K_{2ip} \left(e^x \sqrt{4 \cosh X \cosh Y} \right), \end{aligned} \quad (64)$$

where $E(k) \equiv k^2$, $E(q) \equiv q^2$. This is an inverse integral transform, as in the preceding section, of a generating function

$$\begin{aligned} G(X, Y; x, p) &\equiv \int_{-\infty}^{\infty} \frac{dk}{\sqrt{\sinh(\pi\sqrt{E(k)})}} \int_{-\infty}^{\infty} \frac{dq}{\sqrt{\sinh(\pi\sqrt{E(q)})}} e^{-ikX} e^{-iqY} f_{E(k) E(q)}(x, p) \\ &= \frac{2}{\pi} \left(\frac{\cosh Y}{\cosh X} \right)^{ip} K_{2ip} \left(e^x \sqrt{4 \cosh X \cosh Y} \right) = G^*(Y, X; x, p). \end{aligned} \quad (65)$$

The form and construction of this G are consequences of (61), as detailed in Appendix D.

However, the \star -composition law of this particular generating function is not so straightforward. It is singular, as a consequence of the general relation (24) and the behavior of the integrand in (65) as $k, q \rightarrow 0^1$.

By some contrast to the above, an alternate generating function for just the diagonal WFs, $f_{EE} \equiv f_E$, could be defined through the spectral resolution of the $\star - K$ function,

$$\mathcal{G}(z; x, p) \equiv K_{\star i\sqrt{H(x,p)}}(e^z) = 2\pi \int_0^\infty dE K_{i\sqrt{E}}(e^z) f_E(x, p). \quad (66)$$

This can be evaluated by reliance on Macdonald's trilinear identity [22, 23],

$$\int_0^\infty dE K_{i\sqrt{E}}(e^z) \psi_E(x) \psi_E^*(y) = \frac{1}{2} \exp \left(-\frac{1}{2} (e^{x+y-z} + e^{x-y+z} + e^{-x+y+z}) \right). \quad (67)$$

¹ The singularity may be controlled by regulating the \star -product through imaginary shifts in the momenta,

$$G \left(X, Y; x, p - \frac{i\epsilon}{2} \right) \star G \left(W, Z; x, p + \frac{i\epsilon}{2} \right) = \frac{1}{2\pi} G(W, Y; x, p) \Gamma(\epsilon) \left(e^x \sqrt{\frac{\cosh Y \cosh W}{\cosh X \cosh Z}} (\cosh X + \cosh Z) \right)^{-\epsilon}.$$

It follows that one derivative with respect to either of X or Z suffices to eliminate the divergence at $\epsilon = 0$,

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0} \partial_X G \left(X, Y; x, p - \frac{i\epsilon}{2} \right) \star G \left(W, Z; x, p + \frac{i\epsilon}{2} \right) = \\ &= \frac{1}{2\pi} G(W, Y; x, p) (-\partial_X) \ln \left(e^x \sqrt{\frac{\cosh Y \cosh W}{\cosh X \cosh Z}} (\cosh X + \cosh Z) \right) = \frac{1}{2\pi} G(W, Y; x, p) \left(\frac{1}{2} \tanh X - \frac{\sinh X}{\cosh X + \cosh Z} \right). \end{aligned}$$

Unlike the situation in (58), here the RHS vanishes at $X = Z$. More symmetrically,

$$\lim_{\epsilon \rightarrow 0} \partial_X G \left(X, Y; x, p - \frac{i\epsilon}{2} \right) \star \partial_W G \left(W, Z; x, p + \frac{i\epsilon}{2} \right) = \frac{1}{2\pi} \partial_W G(W, Y; x, p) \left\{ \frac{1}{2} \tanh X - \frac{\sinh X}{\cosh X + \cosh Z} \right\}.$$

\mathcal{G} then is obtained by replacing $x \rightarrow x + Y$ and $y \rightarrow x - Y$, and Fourier transforming by $\frac{1}{\pi} \int dY e^{-2ipY}$,

$$\int_0^\infty dE K_{i\sqrt{E}}(e^z) f_E(x, p) = \frac{1}{2\pi} \int dY e^{-2ipY} \exp\left(-\frac{1}{2}(e^{2x-z} + e^{z+2Y} + e^{z-2Y})\right). \quad (68)$$

Finally, simplifying the RHS gives

$$\begin{aligned} 2\pi \int_0^\infty dE K_{i\sqrt{E}}(e^z) f_E(x, p) &= \exp\left(-\frac{1}{2}e^{2x-z}\right) \int dY e^{-2ipY} \exp\left(-\frac{1}{2}e^z(e^{2Y} + e^{-2Y})\right) \\ &= \exp\left(-\frac{1}{2}e^{2x-z}\right) K_{ip}(e^z) = \mathcal{G}(z; x, p). \end{aligned} \quad (69)$$

As a side check of this expression, (69), note that it must satisfy the equations

$$H \star \mathcal{G}(z; x, p) = \mathcal{G}(z; x, p) \star H = (-\partial_z^2 + e^{2z}) \mathcal{G}(z; x, p), \quad (70)$$

which follows from the spectral resolution evident in (66). Indeed, since $e^{-z}\partial_z K_{ip}(e^z) = ipe^{-z}K_{ip}(e^z) - K_{ip+1}(e^z)$, and $(-\partial_z^2 + e^{2z})K_{ip}(e^z) = p^2 K_{ip}(e^z)$, these relations are satisfied,

$$\begin{aligned} (p^2 + e^{2x}) \star \left(\exp\left(-\frac{1}{2}e^{2x-z}\right) K_{ip}(e^z)\right) &= \left(\exp\left(-\frac{1}{2}e^{2x-z}\right) K_{ip}(e^z)\right) \star (p^2 + e^{2x}) \\ &= \exp\left(-\frac{1}{2}e^{2x-z}\right) (-e^{2x-z} \partial_z K_{ip}(e^z)) + \left(p^2 + \frac{1}{2}e^{2x-z} - \frac{1}{4}e^{4x-2z}\right) \exp\left(-\frac{1}{2}e^{2x-z}\right) K_{ip}(e^z) \\ &= (-\partial_z^2 + e^{2z}) \left(\exp\left(-\frac{1}{2}e^{2x-z}\right) K_{ip}(e^z)\right). \end{aligned} \quad (71)$$

Parenthetically, as an alternative to the ordinary product form in (69), the phase-space kernel \mathcal{G} may also be represented as an integral either of a \star -exponential or of a single \star -product²,

$$\begin{aligned} \mathcal{G}(z; x, p) &= \frac{1}{2} \int dy \exp_\star \left(-\frac{y}{2 \sinh y} e^{2x-z} + iyp - e^z \cosh y \right) \\ &= \frac{1}{2} \int dy \exp \left(-\frac{1}{2} e^{y-z} e^{2x} \right) \star \exp(iyp - e^z \cosh y). \end{aligned} \quad (72)$$

This follows from the identities (cf. Appendix C),

$$\exp_\star \left(-\frac{y}{2 \sinh y} e^{2x-z} + iyp \right) = \exp \left(-\frac{1}{2} e^{y-z} e^{2x} \right) \star \exp(iyp) = \exp \left(-\frac{1}{2} e^{2x-z} + iyp \right). \quad (73)$$

The ordinary product form in (69) and the \star -exponential form in (72) reveal that $\mathcal{G}(z; x, p) = \mathcal{G}(z; x, -p)$, so one may replace $\exp(iyp)$ by $\cos(yp)$ in the second line of (72) above. Given these, there are several ways to verify (70). These relations and the star-product expressions for the kernel in (72) are isomorphic to those of the corresponding operators, as discussed in Appendix E.

The \star -composition law of these generating functions follows from (24) and Macdonald's identity,

$$\mathcal{G}(u; x, p) \star \mathcal{G}(v; x, p) = \frac{1}{2} \int dw \exp \left(-\frac{1}{2} (e^{u+v-w} + e^{u-v+w} + e^{-u+v+w}) \right) \mathcal{G}(w; x, p). \quad (74)$$

² NB Do not shift the integration parameter y by the phase-space variable x before the star products are evaluated.

This also follows directly from the explicit form (69). Again, this is isomorphic to the corresponding operator composition law given in Appendix E.

From the orthogonality of the ψ_{ES} , the diagonal WFs may be recovered by inverse transformation,

$$f_E(x, p) = \int dz \frac{\sinh(\pi\sqrt{E})}{2\pi^3} K_{i\sqrt{E}}(e^z) \mathcal{G}(z; x, p). \quad (75)$$

This representation and the specific factorized x, p -dependence of \mathcal{G} can be of considerable use, e.g., in systematically computing diagonal matrix elements in phase space.

In illustration of the general pattern, consider the first-order energy shift effected by a perturbation Hamiltonian kernel H_1 . It is, cf. Appendix B, eqn (103),

$$\Delta E = \int dz dx dp H_1 \frac{\sinh(\pi\sqrt{E})}{2\pi^3} K_{i\sqrt{E}}(e^z) \mathcal{G}(z; x, p). \quad (76)$$

Choosing

$$H_1 = e^{2nx} e^{isp/2}, \quad (77)$$

yields

$$\Delta E = \frac{\sinh(\pi\sqrt{E})}{2\pi^3} \int dz K_{i\sqrt{E}}(e^z) \left(\int dx e^{2nx} \exp\left(-\frac{1}{2} e^{2x-z}\right) \right) \left(\int dp K_{ip}(e^z) e^{isp/2} \right). \quad (78)$$

Now,

$$\int dx e^{2nx} \exp\left(-\frac{1}{2} e^{2x-z}\right) = 2^{n-1} \Gamma(n) e^{nz}, \quad (79)$$

and hence, ([18] 6.576.4, $a = b$),

$$\begin{aligned} & \int dz K_{i\sqrt{E}}(e^z) K_{ip}(e^z) e^{nz} \\ &= \frac{2^{n-3}}{\Gamma(n)} \Gamma\left(\frac{n+i\sqrt{E}+ip}{2}\right) \Gamma\left(\frac{n+i\sqrt{E}-ip}{2}\right) \Gamma\left(\frac{n-i\sqrt{E}+ip}{2}\right) \Gamma\left(\frac{n-i\sqrt{E}-ip}{2}\right). \end{aligned} \quad (80)$$

Thus,

$$\begin{aligned} \Delta E &= \frac{\sinh(\pi\sqrt{E})}{2\pi^3} 4^{n-2} \times \\ & \int dp e^{isp/2} \Gamma\left(\frac{n+i\sqrt{E}+ip}{2}\right) \Gamma\left(\frac{n+i\sqrt{E}-ip}{2}\right) \Gamma\left(\frac{n-i\sqrt{E}+ip}{2}\right) \Gamma\left(\frac{n-i\sqrt{E}-ip}{2}\right). \end{aligned} \quad (81)$$

Finally, ([18] 6.422.19),

$$\begin{aligned} & \int dp e^{isp/2} \Gamma\left(\frac{n+i\sqrt{E}+ip}{2}\right) \Gamma\left(\frac{n+i\sqrt{E}-ip}{2}\right) \Gamma\left(\frac{n-i\sqrt{E}+ip}{2}\right) \Gamma\left(\frac{n-i\sqrt{E}-ip}{2}\right) \\ &= 4\pi G_{22}^{22} \left(e^s \left| \begin{array}{c} \frac{2-n+i\sqrt{E}}{2}, \frac{2-n-i\sqrt{E}}{2} \\ \frac{n+i\sqrt{E}}{2}, \frac{n-i\sqrt{E}}{2} \end{array} \right. \right). \end{aligned} \quad (82)$$

To sum up, the perturbed energy shift is a Meijer function,

$$\Delta E = \frac{4^n \sinh(\pi\sqrt{E})}{8\pi^2} G_{22}^{22} \left(e^s \left| \begin{array}{c} \frac{2-n+i\sqrt{E}}{2}, \frac{2-n-i\sqrt{E}}{2} \\ \frac{n+i\sqrt{E}}{2}, \frac{n-i\sqrt{E}}{2} \end{array} \right. \right). \quad (83)$$

In principle, any polynomial perturbation in either x or p can be obtained from this, by differentiation with respect to n and s . (Retaining a bit of exponential in x would be helpful to suppress the region of large negative x).

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Appendix A \star -Fock Space and Coherent States

Dirac's Hamiltonian factorization method for algebraic solution of the harmonic oscillator carries through (cf. [2]) intact in \star -space. Indeed,

$$H = \frac{1}{2}(x - ip) \star (x + ip) + \frac{\hbar}{2}, \quad (84)$$

motivating definition of

$$a \equiv \frac{1}{\sqrt{2}}(x + ip), \quad a^\dagger \equiv \frac{1}{\sqrt{2}}(x - ip). \quad (85)$$

Thus, noting

$$a \star a^\dagger - a^\dagger \star a = \hbar, \quad (86)$$

and also that, by above,

$$a \star f_0 = \frac{1}{\sqrt{2}}(x + ip) \star e^{-(x^2+p^2)} = 0, \quad (87)$$

provides a \star -Fock vacuum, it is evident that associativity of the \star -product permits the entire ladder spectrum generation to go through as usual. The \star -genstates of the Hamiltonian, s.t. $H \star f = f \star H$, are thus

$$f_{nn} = f_n = \frac{1}{n!} (a^\dagger \star)^n f_0 (\star a)^n. \quad (88)$$

These states are real, like the Gaussian ground state, and are thus left-right symmetric \star -genstates. They are also transparently \star -orthogonal for different eigenvalues; and they project to themselves, as they should, since the Gaussian ground state does, $f_0 \star f_0 = f_0/2\pi\hbar$.

The complete set of generalized WFs can thus be written as

$$f_{mn} = \frac{1}{\sqrt{n! m!}} (a^\dagger \star)^n f_0 (\star a)^m, \quad m, n = 0, 1, 2, 3, \dots \quad (89)$$

The standard combinatoric features of conventional Fock space apply separately to left and right (its adjoint) \star -multiplication:

$$\begin{aligned}
a \star f_n &\equiv a \star f_{nn} = \hbar \sqrt{n} f_{n,n-1} \\
a^\dagger \star a \star f_n &= \hbar \sqrt{n} a^\dagger \star f_{n,n-1} = \hbar n f_n \\
a^\dagger \star f_n &\equiv a^\dagger \star f_{nn} = \sqrt{n+1} f_{n,n+1} \\
a \star a^\dagger \star f_n &= \sqrt{n+1} a \star f_{n,n+1} = \hbar(n+1) f_n,
\end{aligned} \tag{90}$$

$$\begin{aligned}
f_n \star a &= \sqrt{n+1} f_{n+1,n} \\
f_n \star a \star a^\dagger &= \hbar(n+1) f_n \\
f_n \star a^\dagger &= \hbar \sqrt{n} f_{n-1,n} \\
f_n \star a^\dagger \star a &= \hbar n f_n.
\end{aligned} \tag{91}$$

Furthermore, a left/right (non-self-adjoint) coherent state is naturally defined [19, 24]

$$\Phi(\alpha, \beta) = \exp_\star(\alpha a^\dagger) f_0 \exp_\star(\beta a), \quad a \star \Phi(\alpha, \beta) = \alpha \Phi(\alpha, \beta), \quad \Phi(\alpha, \beta) \star a^\dagger = \beta \Phi(\alpha, \beta). \tag{92}$$

Up to a factor of $\exp((|\alpha|^2 + |\beta|^2)/2)$, this is also the WF of coherent states $|\alpha\rangle$ and $\langle\beta|$, [24]. As indicated in the text, this coherent state is identifiable with the generating function G for the harmonic oscillator.

Appendix B Stationary Perturbation Theory

Perturbation theory could be carried out in Hilbert space and its resulting wavefunctions utilized to evaluate the corresponding WF integrals. However, in the spirit of logical autonomy of Moyal's formulation of Quantum Mechanics in phase space, the perturbed Wigner functions may also be computed ab initio in phase space [17, 25], without reference to the conventional Hilbert space formulation. The basics are summarized below.

As usual, the Hamiltonian kernel decomposes into free and perturbed parts,

$$H = H_0 + \lambda H_1. \tag{93}$$

Fairlie's stationary, real, \star -genvalue equations [8, 4] for the full hamiltonian,

$$H(x, p) \star f_n(x, p) = f_n(x, p) \star H(x, p) = E_n(\lambda) f_n(x, p), \tag{94}$$

are solved upon expansion of their components E and f in powers of λ , the perturbation strength,

$$E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \dots \tag{95}$$

$$f_n = f_n^0 + \lambda f_n^1 + \lambda^2 f_n^2 + \dots. \tag{96}$$

Note the superscripts on E and f are order indices and not exponents. Resolution into individual powers of λ yields the real equations:

$$H_0 \star f_n^0 = f_n^0 \star H_0 = E_n^0 f_n^0 \tag{97}$$

$$H_0 \star f_n^1 + H_1 \star f_n^0 = f_n^1 \star H_0 + f_n^0 \star H_1 = E_n^0 f_n^1 + E_n^1 f_n^0 \tag{98}$$

$$H_0 \star f_n^2 + H_1 \star f_n^1 = f_n^2 \star H_0 + f_n^1 \star H_1 = E_n^0 f_n^2 + E_n^1 f_n^1 + E_n^2 f_n^0. \tag{99}$$

Left multiplication of (98) by $f_n^0 \star$ yields

$$f_n^0 \star H_0 \star f_n^1 + f_n^0 \star H_1 \star f_n^0 = E_n^0 f_n^0 \star f_n^1 + E_n^1 f_n^0 \star f_n^0, \quad (100)$$

and, by (97),

$$f_n^0 \star H_1 \star f_n^0 = E_n^1 f_n^0 \star f_n^0 ; \quad (101)$$

by (11, 10), and the cyclicity of the trace (3),

$$\int dx dp E_n^1 f_n^0 \star f_n^0 = \int dx dp (f_n^0 \star H_1 \star f_n^0) = \int dx dp (H_1 \star f_n^0 \star f_n^0) = \frac{1}{2\pi\hbar} \int dx dp H_1 \star f_n^0 . \quad (102)$$

Hence,

$$E_n^1 = \int dx dp H_1 f_n^0, \quad (103)$$

the diagonal element of the perturbation. For the off-diagonal elements, similarly left- \star -multiply (98) by f_m^0 ,

$$f_m^0 \star H_0 \star f_n^1 + f_m^0 \star H_1 \star f_n^0 = E_n^0 f_m^0 \star f_n^1 + E_n^1 f_m^0 \star f_n^0 . \quad (104)$$

By completeness, f_n^i , $i \neq 0$, resolves to

$$f_n^i = \sum_{k,l} a_{n,kl}^i f_{kl}^0 , \quad (105)$$

the reality condition dictating

$$a_{n,kl}^i = a_{n,lk}^{*i} . \quad (106)$$

Consequently, by 10,

$$E_m^0 \sum_{k,l} a_{n,kl}^1 f_m^0 \star f_{kl}^0 + f_m^0 \star H_1 \star f_n^0 = E_n^0 \sum_{k,l} a_{n,kl}^1 f_m^0 \star f_{kl}^0 + E_n^1 \frac{1}{2\pi\hbar} f_n^0 \delta_{mn} , \quad (107)$$

and hence

$$E_m^0 \sum_k a_{n,km}^1 f_{km}^0 + 2\pi\hbar f_m^0 \star H_1 \star f_n^0 = E_n^0 \sum_k a_{n,km}^1 f_{km}^0 + E_n^1 f_n^0 \delta_{mn} . \quad (108)$$

For $m \neq n$,

$$(E_n^0 - E_m^0) \sum_l a_{n,lm}^1 f_{lm}^0 = 2\pi\hbar (f_m^0 \star H_1 \star f_n^0), \quad (109)$$

so that

$$\sum_l a_{n,lm}^1 f_{lm}^0 = \frac{2\pi\hbar (f_m^0 \star H_1 \star f_n^0)}{E_n^0 - E_m^0} . \quad (110)$$

Finally, use of (11), yields

$$\begin{aligned}
a_{n,lm}^1 &= (2\pi\hbar)^2 \int dx dp \frac{f_{ml}^0 \star f_m^0 \star H_1 \star f_n^0}{E_n^0 - E_m^0} \\
&= (2\pi\hbar)^2 \int dx dp \frac{1}{2\pi\hbar} \frac{f_{ml}^0 \star H_1 \star f_n^0}{E_n^0 - E_m^0} \\
&= (2\pi\hbar) \int dx dp \frac{H_1 \star f_n^0 \star f_{ml}^0}{E_n^0 - E_m^0} \\
&= \frac{\delta_{nl}}{E_n^0 - E_m^0} \int dx dp H_1 f_{mn}^0 \quad (m \neq n) .
\end{aligned} \tag{111}$$

We also have the similar equation for $l \neq n$. Consequently, $a_{n,lm}^1$ is proportional to the matrix element of the perturbation, and it vanishes unless l or m is equal to n . (NB This differs from [25] eqn (45).) To sum up,

$$f_n^1 = \sum_{m \neq n} \frac{1}{E_n^0 - E_m^0} \left(f_{nm}^0 \left(\int dx' dp' H_1(x', p') f_{mn}^0(x', p') \right) + f_{mn}^0 \left(\int dx' dp' H_1(x', p') f_{nm}^0(x', p') \right) \right). \tag{112}$$

By (8), it can be seen that the same result may also follow from evaluation of the WF integrals of perturbed wavefunctions obtained in standard perturbation theory in Hilbert space.

For example, consider $H_1 = \sqrt{2} x = a + a^\dagger$. It follows that $E_0^1 = 0$, and

$$\begin{aligned}
a_{n,lm}^1 &= \frac{\delta_{n,l}}{(E_n^0 - E_m^0)} \iint dx dp f_{mn}^0 \star (a + a^\dagger) \\
&= \frac{\delta_{n,l}}{(E_n^0 - E_m^0)} \iint dx dp (\sqrt{m+1} f_{m+1,n}^0 + \sqrt{n+1} f_{m,n+1}^0) \\
&= \delta_{n,l} (\sqrt{m+1} \delta_{m+1,n} - \sqrt{n+1} \delta_{m,n+1}) ,
\end{aligned} \tag{113}$$

for $m \neq n$ and the $(m \leftrightarrow l)$ expression for $l \neq n$. Hence,

$$f_n^1 = \sqrt{n} (f_{n-1,n}^0 + f_{n,n-1}^0) - \sqrt{n+1} (f_{n,n+1}^0 + f_{n+1,n}^0). \tag{114}$$

Appendix C Combinatoric Derivation of Identities (54) and (73)

The \star -exponential (54) of the Hamiltonian kernel for the linear potential is also easy to work out directly, since the combinatorics in \star -space are identical to the combinatorics of any associative algebra. In particular, the Campbell-Baker-Hausdorff expansion also holds for \star -exponentials,

$$\exp_\star(A) \star \exp_\star(B) = \exp_\star \left(A + B + \frac{1}{2} [A, B]_\star + \frac{1}{12} [A, [A, B]_\star]_\star + \frac{1}{12} [[A, B]_\star, B]_\star + C \right), \tag{115}$$

Where C represents a sum of triple or more nested \star -commutators (Moyal Brackets, $[A, B]_\star \equiv A \star B - B \star A$). Now, choosing $A = itx$ and $B = itp^2 + it^2 p + \frac{1}{3} it^3$, yields $[A, B]_\star = -2it^2 p - it^3$, $[A, [A, B]_\star]_\star = 2it^3$, $[[A, B]_\star, B]_\star = 0$, and hence $C = 0$.

Consequently,

$$\exp_\star(itx) \star \exp_\star \left(itp^2 + it^2 p + \frac{1}{3} it^3 \right) = \exp_\star(itx + itp^2). \tag{116}$$

But further note $\exp_\star(ax) = \exp(ax)$, and also $\exp_\star(bp^2 + cp + d) = \exp(bp^2 + cp + d)$. This reduces the \star -product to a mere translation,

$$\begin{aligned} \exp_\star(ax) \star \exp_\star(bp^2 + cp + d) &= \exp(ax) \star \exp(bp^2 + cp + d) \\ &= \exp\left(ax + \frac{1}{2}ia\partial_p\right) \exp(bp^2 + cp + d) \\ &= \exp\left(ax + b\left(p + \frac{1}{2}ia\right)^2 + c\left(p + \frac{1}{2}ia\right) + d\right) \\ &= \exp\left(ax + bp^2 + (c + iab)p + d - \frac{1}{4}a^2b + \frac{1}{2}iac\right). \end{aligned} \quad (117)$$

Consequently,

$$\exp_\star(itx) \star \exp_\star\left(itp^2 + it^2p + \frac{1}{3}it^3\right) = \exp\left(it\left(x + p^2 + t^2/12\right)\right), \quad (118)$$

and the identity

$$\exp_\star\left(it\left(x + p^2\right)\right) = \exp\left(it\left(x + p^2 + t^2/12\right)\right) \quad (54)$$

follows.

The proof of

$$\exp_\star\left(-\frac{y}{2\sinh y}e^{2x-z} + iyp\right) = \exp\left(-\frac{1}{2}e^{y-z}e^{2x}\right) \star \exp(iyp) = \exp\left(-\frac{1}{2}e^{2x-z} + iyp\right) \quad (73)$$

is similar. Choosing now $A = -\frac{1}{2}e^{y-z}e^{2x}$ and $B = iyp$, it follows that $[A, B]_\star = -2yA$, so that only those multiple Moyal commutators survive which are linear in A . This means, then, that in the Hausdorff expansion [26] for $Z(A, B) \equiv \ln_\star(\exp_\star(A) \star \exp_\star(B))$, only B and terms *linear* in A survive. Hence, Z reduces to merely

$$Z = B + A\left(\frac{B]_\star}{1 - e^{-B]_\star}}\right). \quad (119)$$

The Hadamard expansion in the right parenthesis means successive right \star -commutation with respect to B as many times as the regular power expansion of the function in the parenthesis dictates. Consequently,

$$\exp\left(-\frac{1}{2}e^{y-z}e^{2x}\right) \star \exp(iyp) = \exp_\star\left(-\frac{1}{2}e^{y-z}e^{2x}\right) \star \exp_\star(iyp) = \exp_\star\left(-\frac{y}{2\sinh y}e^{2x-z} + iyp\right). \quad (120)$$

On the other hand,

$$\exp\left(-\frac{1}{2}e^{y-z}e^{2x}\right) \star \exp(iyp) = \exp\left(-\frac{1}{2}e^{y-z+2x}\right) \exp\left(iy\left(p + i\overleftarrow{\partial}_x/2\right)\right) = \exp\left(-\frac{1}{2}e^{2x-z} + iyp\right), \quad (121)$$

and the identity is proven.

Appendix D Construction of the Generating Function for the Liouville WFs

From (60) and (61), it is evident that the Liouville wave functions can be generated by

$$\exp(-e^x \cosh X) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{\sinh(\pi\sqrt{E(k)})}} e^{-ikX} \psi_{E(k)}(x), \quad (122)$$

where $E(k) \equiv k^2$. Therefore, the usual wave function bilinears appearing in the WFs are generated by (recalling that the ψ 's are real)

$$\begin{aligned} & \exp(-e^{x-y} \cosh X) \exp(-e^{x+y} \cosh Y) \\ &= \int_{-\infty}^{\infty} \frac{dk}{\sqrt{\sinh(\pi\sqrt{E(k)})}} \int_{-\infty}^{\infty} \frac{dq}{\sqrt{\sinh(\pi\sqrt{E(q)})}} e^{-ikX-iqY} \psi_{E(k)}(x-y) \psi_{E(q)}(x+y). \end{aligned} \quad (123)$$

Consequently, Fourier transforming this produces a generating function for WFs,

$$\begin{aligned} & \frac{1}{\pi} \int_{-\infty}^{\infty} dy e^{-2ipy} \exp(-e^{x-y} \cosh X) \exp(-e^{x+y} \cosh Y) \\ &= \int_{-\infty}^{\infty} \frac{dk}{\sqrt{\sinh(\pi\sqrt{E(k)})}} \int_{-\infty}^{\infty} \frac{dq}{\sqrt{\sinh(\pi\sqrt{E(q)})}} e^{-ikX-iqY} f_{E(k)E(q)}(x, p). \end{aligned} \quad (124)$$

Evaluation of this expression yields just a factor multiplying a modified Bessel function,

$$\begin{aligned} & \int_{-\infty}^{\infty} dy e^{-2ipy} \exp(-e^{x-y} \cosh X - e^{x+y} \cosh Y) \\ &= \int_{-\infty}^{\infty} dy e^{-2ip(y + \frac{1}{2} \ln(\cosh X / \cosh Y))} \exp\left(-\left(e^x \sqrt{4 \cosh X \cosh Y}\right) \cosh y\right) \\ &= 2 \left(\frac{\cosh Y}{\cosh X}\right)^{ip} K_{2ip}\left(e^x \sqrt{4 \cosh X \cosh Y}\right). \end{aligned} \quad (125)$$

Thus, a generating function for the complete set of Liouville Wigner functions is

$$\begin{aligned} & \frac{2}{\pi} \left(\frac{\cosh Y}{\cosh X}\right)^{ip} K_{2ip}\left(e^x \sqrt{4 \cosh X \cosh Y}\right) \\ &= \int_{-\infty}^{\infty} \frac{dk}{\sqrt{\sinh(\pi\sqrt{E(k)})}} \int_{-\infty}^{\infty} \frac{dq}{\sqrt{\sinh(\pi\sqrt{E(q)})}} e^{-ikX-iqY} f_{E(k)E(q)}(x, p), \end{aligned} \quad (65)$$

as in the text.

Appendix E Operator Ordering and eqn (69)

Given the factorized phase-space generating function

$$\mathcal{G}(z; x, p) = \exp\left(-\frac{1}{2} e^{2x-z}\right) K_{ip}(e^z), \quad (69)$$

what is the operator corresponding to it? According to Weyl's prescription, eq (5), the associated operator is

$$\begin{aligned} \mathfrak{G}(z; \mathcal{X}, \mathcal{P}) &= \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp \mathcal{G}(z; x, p) \exp(i\tau(\mathcal{P} - p) + i\sigma(\mathcal{X} - x)) \\ &= \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp \exp(i\tau\mathcal{P} + i\sigma\mathcal{X}) \exp\left(-\frac{1}{2} e^{2x-z} - i\sigma x\right) K_{ip}(e^z) \exp(-i\tau p). \end{aligned} \quad (126)$$

The integrals over x and p may be evaluated separately, if the σ contour is first shifted slightly above the real axis, $\sigma \rightarrow \sigma + i\epsilon$, thereby suppressing contributions to the x -integral as $x \rightarrow -\infty$. Now $s \equiv \frac{1}{2}e^{2x-z}$ gives

$$\begin{aligned} \int_{-\infty}^{+\infty} dx \exp\left(-\frac{1}{2}e^{2x-z} - i(\sigma + i\epsilon)x\right) &= \int_0^\infty \frac{ds}{2s} (2se^z)^{-i(\sigma+i\epsilon)/2} \exp(-s) \\ &= \frac{1}{2} e^{-i(z+\ln 2)\sigma/2} \Gamma(-i(\sigma + i\epsilon)/2). \end{aligned} \quad (127)$$

By (61),

$$\int dp K_{ip}(e^z) \exp(-i\tau p) = \frac{1}{2} \int_{-\infty}^\infty dX e^{-e^z \cosh X} 2\pi \delta(X - \tau) = \pi e^{-e^z \cosh \tau}. \quad (128)$$

So

$$\mathfrak{G}(z; \mathcal{X}, \mathcal{P}) = \frac{1}{8\pi} \int d\tau d\sigma e^{-i(z+\ln 2)\sigma/2} \Gamma(-i(\sigma + i\epsilon)/2) e^{-e^z \cosh \tau} \exp(i\tau \mathcal{P} + i\sigma \mathcal{X}). \quad (129)$$

The shifted σ contour avoids the pole in Γ at the origin.

Ordering with all \mathcal{P} s to the right, thereby departing from Weyl ordering, yields $\exp(i\tau \mathcal{P} + i\sigma \mathcal{X}) = \exp(i\sigma \mathcal{X}) \exp(i\sigma \tau/2) \exp(i\tau \mathcal{P})$. Performing the σ integration before the τ integration, permits taking the limit $\epsilon \rightarrow 0$ to obtain

$$\begin{aligned} \mathfrak{G}(z; \mathcal{X}, \mathcal{P}) &= \frac{1}{8\pi} \int d\tau \left(\int d\sigma \Gamma(-i(\sigma + i\epsilon)/2) \exp(i\sigma \mathcal{X} + i\sigma \tau/2 - i\sigma(z + \ln 2)/2) \right) e^{-e^z \cosh \tau} \exp(i\tau \mathcal{P}) \\ &= \frac{1}{8\pi} \int d\tau \left(4\pi \exp\left(-e^{2\mathcal{X}+\tau-(z+\ln 2)}\right) \right) e^{-e^z \cosh \tau} \exp(i\tau \mathcal{P}) \\ &= \frac{1}{2} \int d\tau \exp\left(-\frac{1}{2}e^{2\mathcal{X}+\tau-z} - \frac{1}{2}e^{z+\tau} - \frac{1}{2}e^{z-\tau}\right) \exp(i\tau \mathcal{P}). \end{aligned} \quad (130)$$

This is the operator correspondent to (72); it reflects the Weyl correspondence through which it was originally defined (although, technically, it was taken out of Weyl ordering above, merely as a matter of convenience, not a bona-fide change of representation).

This form leads to a more intuitive Hilbert space representation. Acting to the right of a position eigen-bra, $\langle x| \mathcal{X} = \langle x|x$, while the subsequent exponential of the momentum operator just translates, $\langle x| \exp(i\tau \mathcal{P}) = \langle x + \tau|$. So the full right-operation of \mathfrak{G} is

$$\begin{aligned} \langle x| \mathfrak{G}(z; \mathcal{X}, \mathcal{P}) &= \frac{1}{2} \int d\tau \langle x + \tau| \exp\left(-\frac{1}{2}e^{2x+\tau-z} - \frac{1}{2}e^{z+\tau} - \frac{1}{2}e^{z-\tau}\right) \\ &= \frac{1}{2} \int dy \langle y| \exp\left(-\frac{1}{2}e^{x+y-z} - \frac{1}{2}e^{z+y-x} - \frac{1}{2}e^{z-y+x}\right). \end{aligned} \quad (131)$$

Inserting $1 = \int dx |x\rangle \langle x|$ gives $\mathfrak{G}(z; \mathcal{X}, \mathcal{P}) = \int dx |x\rangle \langle x| \mathfrak{G}(z; \mathcal{X}, \mathcal{P})$, and leads to a coordinate space realization of the operator involving an x, y -symmetric kernel,

$$\mathfrak{G}(z; \mathcal{X}, \mathcal{P}) = \frac{1}{2} \int dx dy |x\rangle \langle y| \exp\left(-\frac{1}{2}e^{x+y-z} - \frac{1}{2}e^{x-y+z} - \frac{1}{2}e^{-x+y+z}\right). \quad (132)$$

This operator is diagonal on energy states: by Macdonald's identity (67), and the reality and orthogonality of the wave functions,

$$\begin{aligned}\langle E_1 | \mathfrak{G}(z; \mathcal{X}, \mathcal{P}) | E_2 \rangle &= \frac{1}{2} \int dx dy \psi_{E_1}^*(x) \psi_{E_2}(y) \exp \left(-\frac{1}{2} e^{x+y-z} - \frac{1}{2} e^{z+y-x} - \frac{1}{2} e^{z-y+x} \right) \\ &= \delta(E_1 - E_2) K_{i\sqrt{E_1}}(e^z) .\end{aligned}\quad (133)$$

This is in agreement with the corresponding phase-space expression, (66).

The composition law of this operator also parallels its phase-space isomorph, (74),

$$\mathfrak{G}(u)\mathfrak{G}(v) = \frac{1}{2} \int dw \exp \left(-\frac{1}{2} (e^{u+v-w} + e^{u-v+w} + e^{-u+v+w}) \right) \mathfrak{G}(w). \quad (134)$$

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